Two-loop approximation in the Coulomb blockade problem

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We study Coulomb blockade (CB) oscillations in the thermodynamics of a metallic grain which is connected to a lead by a tunneling contact with a large conductance g_0 in a wide temperature range, $E_C g_0^4 e^{-g_0/2} < T < E_C$, where E_C is the charging energy. Using the instanton analysis and the renormalization group we obtain the temperature dependence of the amplitude of CB oscillations which differs from the previously obtained results. Assuming that at $T < E_C g_0^4 e^{-g_0/2}$ the oscillation amplitude weakly depends on temperature we estimate the magnitude of CB oscillations in the ground state energy as $E_C g_0^4 e^{-g_0/2}$.

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I. INTRODUCTION AND MAIN RESULTS

The study of electron-electron interactions in mesoscopic systems has been at the focus of experimental and theoretical interest over the past two decades. One of the most striking consequences of electron interactions at low temperatures is the phenomenon of Coulomb blockade.[1] For example, thermodynamic quantities of a metallic grain which is connected by a tunneling contact to a metallic lead and is capacitively coupled to a gate exhibit oscillatory dependence on the gate voltage. This can be observed experimentally by measuring the differential capacitance of the grain.[2]

At low temperatures the Coulomb interaction of electrons in the grain can be described within the framework of the constant interaction model

$$\hat{H}_C = E_C \left(\hat{N} - q\right)^2, \tag{1}$$

where E_C is the capacitive charging energy of the grain, $\hat{N}$ is the operator of the number of electrons in it, and q is the dimensionless gate voltage.

For a grain with a vanishing mean level spacing and for a tunneling contact with a large dimensionless conductance $g_0 = 2\pi\hbar/e^2 R \gg 1$, where R is the resistance of the contact, the amplitude of Coulomb blockade oscillations in the ground state energy is exponentially small,[3] $\propto E_C \exp(-\frac{g_0}{2})$. The magnitude and temperature dependence of Coulomb blockade oscillations in this problem were studied in many theoretical works using the renormalization group (RG) treatment[4,5] and the instanton[6] approach[7,8] to the dissipative action of Ref. 9.

In the instanton approach the cost of one instanton is $\propto \exp(-\frac{g(T)}{2})$, where $g(T) \approx g_0 - 2\ln\frac{g_0 E_C}{2\pi^2 T}$,[4] is the renormalized conductance. The instanton gas can be considered as non-interacting at relatively high temperatures, when the renormalized conductance $g(T)$ is still large. In this regime the thermodynamic potential depends sinusoidally on the gate voltage,[8,10]

$$\Omega_{osc}(q) = -\tilde{E}_C(T)\cos(2\pi q), \tag{2}$$

where $\tilde{E}_C(T)$ is the renormalized temperature-dependent charging energy.

Integration over the massive fluctuations around the instanton in the Gaussian approximation leads to the logarithmic temperature dependence[8] of the renormalized charging energy at relatively high temperature. This results in the estimate $E_C g_0^3 \exp(-\frac{g_0}{2})$ for the amplitude of the Coulomb blockade oscillations in ground state energy.

In this paper we evaluate non-Gaussian corrections for the fluctuations about the instanton configurations in the lowest order of perturbation theory in $1/g_0$. We show that these corrections diverge at $T \to 0$ and significantly modify the magnitude and the temperature dependence of the preexponential factor of the Coulomb blockade oscillations even at relatively high temperatures, where the renormalized conductance $g(T)$ is large and non-interacting instanton gas approximation is still valid. We then apply a two-loop renormalization group (RG) to determine the preexponential factor in the renormalized charging energy beyond the region of applicability of perturbation theory.

The main result of this paper is the following expression for the renormalized charging energy in Eq. (2),

$$\tilde{E}_C(T) = \frac{E_C g_0^{5/2}}{3\pi^2 g(T)}\left[g^{3/2}(T_0) - g^{3/2}(T)\right] e^{-g_0/2}. \tag{3a}$$

Here $g(T)$ is the renormalized temperature dependent conductance, and $T_0 = \frac{E_C}{2\pi^2}$. With sufficient accuracy $g(T_0)$ can be found using the perturbation theory, $g(T_0) = g_0 - 2\ln g_0$. At lower temperatures, but still such that $g(T) \gg 1$ the renormalized conductance $g(T)$ can be found from the implicit relation,

$$g(T) = g_0 - 2\ln\frac{g_0 T_0}{T} + 2\ln\frac{g(T)}{g_0}, \tag{3b}$$

which solves the two-loop renormalization group equation.[5]

Equations (2) and (3) rely on the non-interacting instanton gas approximation and are valid at $T > E_C g_0^4 e^{-g_0/2}$. Assuming that at lower temperatures the amplitude of Coulomb blockade oscillations is only weakly temperature-dependent we obtain the estimate for the magnitude of oscillations in the ground state energy, $E_C g_0^4 e^{-g_0/2}$. This is by a factor of g_0 greater than the result of Ref. 8.

The paper is organized as follows: In Sec. II we describe the dissipative action approach to the problem. In Sec. III we discuss the instanton gas approximation. In Sec. III A we treat the fluctuations around the instantons in the Gaussian approximation, and in Sec. III B we obtain the leading $1/g_0$ correction to the Gaussian result. In Sec. IV we use the two-loop renormalization group to extend the perturbative results of Sec. III B to the low temperature regime $\ln(E_C/T) \sim g_0$. Our results are summarized in Sec. V.

II. PARTITION FUNCTION

Following Ref. 9 the partition function of the system can be written as an imaginary time functional integral over the auxiliary field $\phi(\tau)$ which decouples the Coulomb interaction Eq. (1). Different configurations of the field $\phi(\tau)$ fall into different topological classes labeled by the winding number W, characterizing the boundary conditions in imaginary time: $\phi_W(\tau+\beta) = \phi_W(\tau) + 2\pi W$, where $\beta = 1/T$. The partition function is then written as a sum over the winding numbers,

$$Z(q) = \sum_{W=-\infty}^{+\infty} e^{2\pi i q W} Z_W, \tag{4}$$

where Z_W is a functional over the fields ϕ_W in the topological class W given by

$$Z_W = \int D[\phi_W] e^{-S[\phi_W]}. \tag{5}$$

The action $S[\phi]$ has the following form,

$$S[\phi] = S_d[\phi] + \int_0^\beta \frac{\dot\phi^2(\tau)}{4E_C} d\tau. \tag{6}$$

The second term in the right hand side of Eq. (6) is the charging energy term. The first term describing the tunnel junction was obtained in Ref. 9 and can be conveniently written using the complex variable $u = \exp(2\pi i T\tau)$ as follows,

$$S_d[\phi] = -g_0 \oint \frac{du du_1}{(2\pi i)^2} \frac{\mathrm{Re}\left(1 - \mathrm{e}^{\mathrm{i}\phi(\mathrm{u}) - \mathrm{i}\phi(\mathrm{u}_1)}\right)}{(u-u_1)^2}. \tag{7}$$

The integral in this equation is taken over the unit circle, $|u|, |u_1| = 1$.

The minimum of the dissipative action (7) in the topological sector W is equal to $g_0|W|/2$. In the trivial topological sector, $W=0$, it is achieved when $\phi(\tau) = 0$. In the sectors $W = \pm 1$ it is achieved on the instanton configurations[6] which can be written in the complex notations as[11]

$$\exp(i\phi_z) = f(u) = \frac{u - z}{1 - u z^*}, \tag{8}$$

where z (with $|z| < 1$ for $W = 1$ and $|z| > 1$ for $W = -1$) is a complex number characterizing the position and the width of the instanton. In the topological sector with $|W| > 1$ the dissipative action (7) is minimized on the multi-instanton configuration of the field ϕ given by the product of $|W|$ single-instanton terms, as in the right hand side of Eq. (8).

The topological sector with $W = 0$ in Eq. (4) does not contribute to the oscillatory part of the thermodynamic potential, $\Omega(q) = -T \ln Z(q)$. At relatively high temperatures, when the renormalized conductance is large, $g \gg 1$, the main non-zero contribution to the oscillatory part of $\Omega(q)$ comes from the terms with winding numbers $W = \pm 1$ in Eq. (4). All other terms in Eq. (4) are exponentially small in g, and the renormalized charging energy in Eq. (2) can be expressed as

$$\tilde E_C(T) = 2T \frac{Z_1}{Z_0}. \tag{9}$$

We therefore concentrate below on the topological sector $W = 1$.

III. SINGLE INSTANTON APPROXIMATION

Since the value of the dissipative action on the instanton configuration ϕ_z in Eq. (8) is independent of the instanton parameter z it is convenient to write the fields in the topological sector $W = 1$ in the form

$$\phi_1 = \phi_z + \tilde\phi_z, \tag{10}$$

where $\tilde\phi_z$ are massive fluctuations which are orthogonal to the two zero modes of the dissipative action, $\partial\phi_z/\partial z$ and $\partial\phi_z/\partial z^*$. The renormalized charging energy (9) can be written as

$$\tilde E_C(T) = 2T \int \frac{d^2 z}{1 - |z|^2} Z_1(z), \tag{11}$$

where $Z_1(z)$ is given by the following ratio of functional integrals over the massive modes only,

$$Z_1(z) = \frac{\int D[\tilde\phi_z](1-|z|^2) J(z,\tilde\phi_z) \exp(-S[\phi_z + \tilde\phi_z])}{\int D[\phi_0] \exp(-S[\phi_0])}, \tag{12}$$

where $J(z, \tilde\phi_z)$ the Jacobian of the variable transformation, Eq. (10).

Below we compute the renormalized charging energy $\tilde{E}_C$, Eq. (11). In section III A we evaluate the functional integral over the massive modes (12) in the Gaussian approximation. In section III B we obtain corrections to the Gaussian approximation in leading order of perturbation theory in $1/g_0$. Then, in section IV we employ the renormalization group to obtain $\tilde{E}_C$ beyond the regime of validity of perturbation theory.

A. Gaussian approximation

To leading order in $1/g_0$ we may evaluate the ratio of the functional integrals in Eq. (12) in the Gaussian approximation. To this end we expand the actions in the numerator and in the denominator in Eq. (12) to second order in $\tilde{\phi}$ and ϕ_0 respectively.

In the Matsubara basis,

$$\phi_0 = \sum_{n=-\infty}^{\infty} \varphi_n u^n, \tag{13}$$

the action in the denominator acquires the following diagonal form,

$$S^0_{\phi^2} = g_0 \sum_{n\geq 1}^{\infty} (n + an^2)|\varphi_n|^2, \tag{14}$$

where we introduced the notation $a = \frac{2\pi^2 T}{g_0 E_C}$.

To evaluate the functional integral in the numerator in Eq. (12) it is convenient to expand the massive fluctuations $\tilde{\phi}_z$ using the basis of Ref. 6 which in the complex notations[11] can be written as,

$$\tilde{\phi}_z = \sum_{n>0} \tilde{\varphi}_n u^n f(u) + \sum_{n<0} \tilde{\varphi}_n u^n f^*(u). \tag{15}$$

Here $f(u)$ is defined in Eq. (8). In this basis to second order in $\tilde{\phi}_z$ the dissipative part of the action, Eq. (7), has the following diagonal form,

$$S^i_{\tilde{\phi}^2} = \frac{g_0}{2} + g_0 \sum_{n\geq 1}^{\infty} n|\tilde{\varphi}_n|^2. \tag{16}$$

The superscript i here refers to the presence of the instanton, whereas the superscript 0 in Eq. (14) denotes the trivial topological sector, $W = 0$.

Instead of calculating the Jacobian $J(z, \tilde{\phi}_z)$ in Eq. (12) directly, we can express the integration measure through the metric tensor $\hat{A}(z, \tilde{\phi}_z)$ using the identity[12]

$$J(z, \tilde{\phi}_z) D[\tilde{\phi}_z] = \sqrt{\det \hat{\mathrm{A}}(\mathrm{z}, \tilde{\phi}_\mathrm{z})} \prod_n d\tilde{\varphi}_n. \tag{17}$$

The metric tensor $\hat{A}(z, \tilde{\phi}_z)$ is presented in Appendix A. Fortunately we do not need to evaluate its determinant for an arbitrary configuration of the field $\tilde{\phi}_z$. Indeed, the fluctuations of the massive modes, $\tilde{\varphi}_n$ are small as $1/g_0$, see Eq. (16). Therefore, for large g_0 we can expand the determinant of the metric tensor in the powers of $\tilde{\varphi}_n$. In leading order in $1/g_0$, i.e. in the Gaussian approximation it is sufficient to evaluate $\det \hat{\mathrm{A}}(\mathrm{z}, \tilde{\phi}_\mathrm{z})$ on the instanton configuration, $\tilde{\varphi}_n = 0$. We show in Appendix A that

$$J(z, 0) = \sqrt{\det \hat{\mathrm{A}}(\mathrm{z}, 0)} = \frac{1}{1 - |z|^2}. \tag{18}$$

The quadratic form of the charging part of the action given by the second term in Eq. (6) is not diagonal in the basis (15). However, for the purpose of evaluating the functional integral in the numerator of Eq. (12) one can neglect its off-diagonal elements. We show in Appendix B that their contribution is small as $1/g_0$. The diagonal part of the charging action is given by Eq. (B6). Then, the ratio of the functional integrals in (12) reduces to the product

$$\frac{g_0}{\pi} \prod_{n=2}^{\infty} \frac{n + an^2}{n - 1 + an^2 + \frac{2a|z|^2}{1-|z|^2}} = \frac{g_0}{\pi} a^{-1 + \frac{2a|z|^2}{1-|z|^2}}, \tag{19}$$

where a was defined below Eq. (14). Since the characteristic instanton frequencies are $T/(1-|z|^2) \leq E_C \ll g_0 E_C$ the second term in the exponent of this expression can be neglected. Using the expression Eq. (B2) for the charging action on the instanton configuration we obtain in the Gaussian approximation,

$$Z_1^{\mathrm{G}}(z) = \frac{g_0^2 E_C}{2\pi^3 T} \exp\left(-\frac{g_0}{2} - \frac{\pi^2 T [1 + |z|^2]}{E_C [1 - |z|^2]} \right). \tag{20}$$

Substituting this expression into Eq. (11) we obtain with logarithmic accuracy,

$$\tilde{E}_C(T) = \frac{g_0^2 E_C}{\pi^2} \ln\left[\frac{T_0}{T}\right] \exp\left(-\frac{g_0}{2}\right), \tag{21}$$

where we introduced the notation

$$T_0 = \frac{E_C}{2\pi^2}. \tag{22}$$

Equation (21) coincides with the result of Ref. 8. One can neglect the interaction between instantons and obtain Eq. (21) only when $Z_1(z) \ll 1$. In section III B we show that even in this regime the non-Gaussian corrections to the functional integral in Eq. (12) lead to large corrections to the preexponential factor in Eq. (21).

B. Corrections to the Gaussian approximation

The Gaussian approximation discussed in section III A is asymptotically exact at $g_0 \to \infty$. The corrections to it are small in $1/g_0$ and may be evaluated perturbatively. To obtain the leading $1/g_0$ correction it is sufficient to expand the Jacobian in Eq. (12) to second order in $\tilde{\phi}_z$

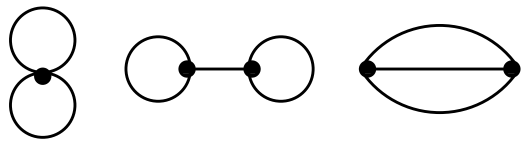

FIG. 1: Diagrams representing $1/g_0$ corrections to $\frac{Z_1(z)}{Z_1^{\rm G}(z)}$ in Eq. (25). The vertices are proportional to g_0, and the Gaussian propagators of fields ϕ depicted by the solid lines are proportional to $1/g_0$.

and to expand the actions in the numerator and in the denominator in Eq. (12) to fourth order in $\tilde\phi_z$ and ϕ_0 respectively:

$$S_d[\phi_1] = S^i_{\tilde\phi^2} + S^i_{\tilde\phi^3} + S^i_{\tilde\phi^4}, \quad (23a)$$

$$S_d[\phi_0] = S^0_{\phi^2} + S^0_{\phi^4}. \quad (23b)$$

The terms $S^0_{\phi^2}$ and $S^i_{\tilde\phi^2}$ were defined in Eqs. (14) and (16) respectively, and $S^i_{\tilde\phi^3}$, $S^i_{\tilde\phi^4}$, and $S^0_{\phi^4}$ are given by the following equations;

$$S^i_{\tilde\phi^4} = \frac{g_0}{24}\oint \frac{du du_1}{(2\pi i)^2}{\rm Re}\left[\frac{f(u)}{f(u_1)}\right]\frac{(\tilde\phi-\tilde\phi_1)^4}{(u-u_1)^2}, \quad (24a)$$

$$S^i_{\tilde\phi^3} = \frac{g_0}{6}\oint \frac{du du_1}{(2\pi)^2}{\rm Im}\left[\frac{f(u)}{f(u_1)}\right]\frac{(\tilde\phi-\tilde\phi_1)^3}{(u-u_1)^2}, \quad (24b)$$

$$S^0_{\phi^4} = \frac{g_0}{24}\oint \frac{du du_1}{(2\pi i)^2}\frac{(\phi-\phi_1)^4}{(u-u_1)^2}. \quad (24c)$$

Here the function $f(u_i)$ was defined in Eq. (8), and we introduced the short hand notations $\tilde\phi_i = \tilde\phi(u_i)$. Substituting Eqs. (23) into Eq. (12) and using Eqs. (17) and (18) we obtain up to terms of order $1/g_0$,

$$\frac{Z_1(z)}{Z_1^{\rm G}(z)} = (1-|z|^2)\left\langle\sqrt{\det\hat A(z,\tilde\phi_z)}\right\rangle_i - \left\langle S^i_{\tilde\phi^4}\right\rangle_i + \left\langle S^0_{\phi^4}\right\rangle_0 + \frac{1}{2}\left\langle (S^i_{\tilde\phi^3})^2\right\rangle_i, \quad (25)$$

where $\langle\ldots\rangle_i$ and $\langle\ldots\rangle_0$ denote averaging with respect to the Gaussian actions in the presence and in the absence of the instanton respectively.

The first term in the right hand side of Eq. (25) needs to be evaluated to second order in $\tilde\phi_z$. This is carried out in Appendix A. The result is given by Eq. (A4). The calculation of the other terms is facilitated by the use of Wick's theorem and reduces to evaluating the diagrams in Fig. 1. The calculations are straightforward and are presented in Appendix C. Only fluctuations of ϕ with relatively low frequencies, $\leq T/(1-|z|^2)$, contribute to the corresponding functional integrals. We can therefore neglect the charging energy term in the Gaussian action for the massive modes. The calculations are further simplified by observing that the two-point correlation functions in the diagrams in Fig. 1 become diagonal in the bases (13) and (15). We then obtain with logarithmic accuracy,

$$Z_1(z) = Z_1^{\rm G}(z)\left[1+\frac{1}{g_0}\ln\frac{g_0T_0}{T} - \frac{1}{g_0}\ln(1-|z|^2)\right], \quad (26)$$

where T_0 was defined in Eq. (22). Substituting Eqs. (26) and (20) into (11) and using Eq. (A4) we obtain for the renormalized charging energy,

$$\tilde E_C(T) = \frac{g_0^2E_C}{\pi^2}\ln\frac{T_0}{T}\left[1+\frac{\ln\frac{g_0T_0}{T}}{g_0}+\frac{\ln\frac{T_0}{T}}{2g_0}\right]e^{-g_0/2}. \quad (27)$$

Equations (26) and (27) represent the main results of this section. The first term in the right hand side of Eq. (27) coincides with the result of Ref. 8, and the others represent a perturbative correction arising from taking into account non-Gaussian fluctuations. Equation (27) is valid at relatively high temperatures, $\ln\frac{T_0}{T}\ll g_0$, when the non-Gaussian correction is small. We consider the region of lower temperatures, $\ln\frac{T_0}{T}\sim g_0$, in section IV using the renormalization group approach.

IV. RENORMALIZATION GROUP

At low temperatures, when $\ln\frac{g_0T_0}{T}$ becomes of the order of g_0, the non-Gaussian correction in Eq. (26) becomes significant, and the perturbative expressions (26) and (27) are no longer valid. There is a wide temperature range in which the perturbative approach used in section III fails but the renormalized conductance is still large, $g(T)\approx g_0 - 2\ln\frac{g_0T_0}{T}\gg 1$, and therefore the single instanton approximation, Eq. (11) is still valid. Below we apply the renormalization group to study the amplitude of the Coulomb blockade oscillations in this regime.

From Eq. (20) we observe that $Z_1(z)$ depends exponentially on the renormalized conductance, $Z_1(z)\propto\exp(-g(T)/2)$. Therefore to obtain the preexponential factor in the renormalized charging energy in Eq. (11) it is necessary to compute the renormalized conductance using the two-loop renormalization group.

We obtain and solve the two-loop RG equations for the conductance in Sec. IV A. Then, in Sec. IV B we obtain the renormalized charging energy $\tilde E_C$, Eq. (11), by evaluating the functional integral $Z_1(z)$ in Eq. (12) with the aid of the renormalization group.

A. Renormalized conductance

To obtain the two-loop RG equations we expand the dissipative action in Eq. (7) to sixth order in ϕ_0. Next we introduce a running frequency scale ν and write $\phi_0 = \phi_0^s + \phi_0^f$. Here ϕ_0^f represents fast degrees of freedom with Matsubara frequencies $\nu_n = Tn$ satisfying the inequality $\nu < \nu_n < g_0T_0$, and ϕ_0^s represents slow degrees of freedom with $\nu_n < \nu$. The calculation of the renormalized conductance at frequency ω amounts to evaluating

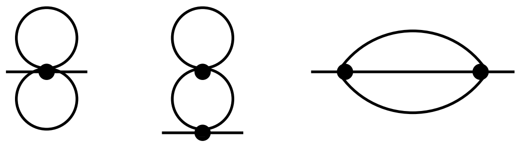

FIG. 2: Diagrams of order $1/g_0$ for the renormalized conductance $g(\nu)$.

the diagrams in Fig. 2, where internal lines correspond to the propagators of the fast degrees of freedom.

The role of the charging term in the Gaussian action, Eq. (14), amounts merely to the ultraviolet cutoff of the frequency integrals over the internal lines at frequencies g_0T_0, where T_0 was defined in Eq. (22). As a result we obtain,

$$g(\nu) = g_0 - 2\ln\frac{g_0T_0}{\nu} - \frac{4}{g_0}\ln\frac{g_0T_0}{\nu}. \tag{28}$$

From Eq. (28) we obtain the following two-loop renormalization group equation for the conductance in agreement with Ref. 5,

$$\frac{dg(\nu)}{d\xi} = -2 - \frac{4}{g(\nu)}, \qquad \xi = \ln\frac{\Lambda}{\nu}, \tag{29}$$

where $\Lambda = g_0T_0$ is an ultraviolet cutoff. The first term in the right hand side of Eq. (29) reproduces the one-loop renormalization group equation of Refs. 4,13.

By integrating the two-loop renormalization group equation, Eq. (29) from $\nu = g_0T_0$ to $\nu = T$ we obtain Eq. (3b) which implicitly determines the temperature dependence of the renormalized conductance $g(T)$.

B. Renormalized charging energy

Next, we apply the renormalization group to evaluate $Z_1(z)$ in Eq. (12). We start by considering the case of the so-called line instantons, $z = 0$. According to the RG procedure the functional integration is performed by separating ϕ into fast and slow degrees of freedom, $\phi = \phi^f + \phi^s$, where ϕ^f includes fluctuations with Matsubara frequencies ν_n in a narrow interval, $\nu = \Lambda e^{-\xi} < \nu_n < \Lambda$. Upon the integration over the fast degrees of freedom the remaining functional integral acquires a factor resulting from integrating out the fast fluctuations of ϕ. This factor depends on the running coupling constant $g(\nu)$. The remaining action for the slow fluctuations is characterized by a renormalized conductance. Due to this multiplicative nature of the functional integration we can express the logarithmic derivative of $Z_1(0)$ through a function of the renormalized conductance $g(\nu)$ only, $\frac{d\ln Z_1(0)}{d\xi} = f[g(\nu)]$. From the perturbative results (26) and (20) we obtain for the logarithmic derivative,

$$\frac{d\ln Z_1(0)}{d\xi} = 1 + \frac{1}{g(\nu)}. \tag{30}$$

Using the two-loop renormalization group for the conductance, Eq. (29) we obtain,

$$\frac{d\ln Z_1(0)}{dg(T)} = -\frac{1}{2} + \frac{1}{2g(T)}. \tag{31}$$

The solution of this equation which satisfies the high temperature asymptotic (26), (20) is given by,

$$Z_1(0) = \frac{\sqrt{g_0 g(T)}}{2\pi^3}\exp\left[-\frac{g(T)}{2}\right]. \tag{32}$$

One can directly check this expression against the perturbative result Eq. (26). To obtain $Z_1(0)$ to order $1/g_0$ we need to evaluate $g(T)$ in the exponent of this equation to the two-loop order, and $g(T)$ in the preexponential factor to the one-loop order only.

To evaluate $Z_1(z)$ at finite z it is convenient to express it in the form

$$Z_1(z) = Z_1(0)X(z)\exp\left(-\frac{\pi^2T[1+|z|^2]}{E_C[1-|z|^2]}\right), \tag{33}$$

where the charging energy on the instanton configuration is written out explicitly, c.f. Eq. (20).

The perturbative result (26) implies $X(z) = 1 - \frac{1}{g_0}\ln(1-|z|^2)$, where the logarithmic term arises from the integration over the fluctuations of ϕ with frequencies ranging between T and the instanton frequency $\nu_z = T/(1-|z|^2)$, see the discussions above Eq. (26) and in Appendix C. Therefore in the renormalization group treatment the conductance g_0 in the logarithmic correction should be understood as the renormalized conductance $g(\nu_z) \approx g(T) + 2\ln\frac{\nu_z}{T}$ at the instanton frequency ν_z. Using the considerations presented above Eq. (32) and the perturbative result, Eq. (26) we express the logarithmic derivative of $X(z)$ through the renormalized conductance $g(\nu_z)$ as,

$$\frac{d\ln[X(z)]}{d\ln\nu_z} = \frac{1}{g(\nu_z)}. \tag{34}$$

Using the one-loop renormalization group equation $\frac{dg(\nu_z)}{d\ln\nu_z} = 2$ and the boundary condition $X(0) = 1$ we find,

$$X(z) = \sqrt{\frac{g(\nu_z)}{g(T)}}. \tag{35}$$

Substituting Eq. (33) into Eq. (11) we write the renormalized charging energy as

$$\tilde{E}_C(T) = 2TZ_1(0)\int\frac{d^2z}{1-|z|^2}X(z)\exp\left[-\frac{\pi^2T[1+|z|^2]}{E_C[1-|z|^2]}\right].$$

Next, we express the integration measure through the differential of the renormalized conductance, $dg(\nu_z)$, according to the one-loop RG equation, $\frac{d^2z}{1-|z|^2} = -\pi d\ln(1-|z|^2) = \frac{\pi}{2}dg(\nu_z)$. The resulting integral over the $dg(\omega_z)$

ranges from $g(\omega_z) = g(T)$ to $g(\nu_z) = g(T_0) = g_0 - 2\ln g_0$. Using Eqs. (32) and (35) we obtain for the renormalized charging energy,

$$\tilde{E}_C(T) = \frac{T\sqrt{g_0}}{3\pi^2}\left[g^{3/2}(T_0) - g^{3/2}(T)\right]e^{-g(T)/2}, \quad (36)$$

where T_0 was defined in Eq. (22).

Next we express the temperature T through the renormalized conductance $g(T)$ using the solution of the two-loop RG equation, Eq. (3b). Substituting this result into Eq. (36) we obtain Eq. (3a).

Equations (3) represent the main result of this paper. They determine the temperature dependence of the amplitude of Coulomb blockade oscillations in a wide temperature regime where the renormalized conductance $g(T)$ is large and the non-interacting instanton gas approximation[8] is valid.

At relatively high temperatures, when $\ln\frac{g_0T_0}{T} \ll g_0$, Eq. (3) reproduces the perturbative result, Eq. (27). At lower temperatures, when $\ln\frac{g_0T_0}{T} \sim g_0$, the renormalized charging energy in Eq. (3a) significantly exceeds the prediction of the Gaussian approximation[8], Eq. (20).

The non-interacting instanton gas approximation is valid while $Z_1/Z_0 < 1$. Let us extrapolate the results (3) to the low temperature limit, when $Z_1/Z_0 \sim 1$. This happens at $T \sim E_C g_0^4 e^{-g_0/2}$ when $g(T) \approx 4\ln g_0$. Substituting this conductance into Eq. (3a) we obtain the following estimate for the amplitude of Coulomb blockade oscillations in the ground state energy, $\tilde{E}_C \sim E_C g_0^4 e^{-g_0/2}$. This exceeds the estimate arising from the Gaussian treatment of the fluctuations[8] by a large factor, g_0.

V. DISCUSSION

A. Summary of the results

We studied the amplitude, $\tilde{E}_C(T)$, of the Coulomb blockade oscillations in the thermodynamic potential for a metallic grain connected to a lead by a tunneling contact with a large tunneling conductance g_0. The effects of a finite mean level spacing in the grain[14] were neglected. We worked withing the non-interacting instanton gas approximation. By applying perturbation theory we obtained the leading $1/g_0$ correction to the Gaussian result[8] for the renormalized charging energy, Eq. (27). Combining the instanton analysis with the two-loop renormalization group we found the temperature dependence of the renormalized charging energy $\tilde{E}_C(T)$, and of the renormalized conductance $g(T)$, Eq. (3), in a wide temperature range $E_C g_0^4 e^{-g_0/2} < T < E_C$. The use of the two-loop RG enables us to determine the preexponential factor in the temperature dependence of $\tilde{E}_C(T)$ even at relatively low temperatures, when $\ln\frac{g_0T_0}{T} \sim g_0$. In this regime the renormalized charging energy in Eq. (3a) is significantly greater than the result of the Gaussian approximation, Eq. (21).

Assuming that the amplitude of the Coulomb blockade oscillations weakly depends on temperature at $T < E_C g_0^4 e^{-g_0/2}$ from Eq. (3) we obtain the following estimate for the amplitude of the oscillations in the ground state energy, $\tilde{E}_C \sim E_C g_0^4 e^{-g_0/2}$. This estimate is greater than the result of the Gaussian approximation[8] by a factor of g_0. In Ref. 7 the result for the integral over the massive fluctuations around the instanton configuration differed from that in Ref. 8 and in Eq. (20). This lead to a different estimate for the amplitude of the Coulomb blockade oscillations in the ground state energy, $E_C g_0^2 e^{-g_0/2}$. In Ref. 5 the instanton effects were ignored, and the estimate $E_C g_0 e^{-g_0/2}$ for the amplitude of the ground state energy oscillations was obtained using the two-loop renormalization group. The topological effects, on the other hand, may drastically alter the results of a perturbative RG consideration. This is well known, for example, in the theory of antiferromagnetic Heisenberg spin chains,[15] where the perturbative RG predicts the appearance of a spin gap for an arbitrary value of the spin S. However, for half-integer spin the topological effects are expected to lead to a gapless state.[16]

B. Applicability of the results and comparison with numerical studies

Within the framework of the model considered here the thermodynamics of the grain is described by Eqs. (4)-(7). The partition function, Eq. (4), depends on two dimensionless parameters, g_0 and T/E_C. This model has been studied numerically by several groups.[17,18]

To compare our results with the numerical studies it is important to keep in mind the approximations that were made. These approximations determine the region of applicability of the instanton approach in the space of parameters g_0 and T/E_C. Below we review the approximations made:

i) The instanton configuration was found by neglecting the charging part of the action in comparison with the dissipative part. This is justified if $a = \frac{2\pi^2T}{g_0E_C} = \frac{T}{g_0T_0} \ll 1$.

ii) To obtain the Gaussian result, Eq. (21), we assumed that the charging action on the instanton configuration, second term in the exponent in Eq. (20), is negligible for long instantons, $z \to 0$. This is valid for $T \ll T_0$. In addition, Eq. (21) holds with logarithmic accuracy. This implies not only that $\frac{T_0}{T} \gg 1$ but that $\ln\frac{T_0}{T} \gg 1$. The low temperature regime is difficult to study numerically. At the lowest available temperature in Ref. 17, $\ln\frac{T_0}{T} = 3.2$. Thus, even at the lowest temperature, the result of Ref. 7 is not parametrically different from that of Ref. 8, Eq. (21).

iii) The use of the single instanton approximation imposes the lower bound, T^*, on the temperature range of applicability of the results. In Ref. 10 the two-instanton contribution to the partition function was taken into account in the one-loop approximation. It was shown that the single instanton approximation, Eq. (9), applies as

long as $\frac{Z_1}{Z_0} \ll 1$, i.e. for $T^* \gg \tilde{E}_C(T^*)$. It is easy to see from Eq. (21) that T^* decreases as the dimensionless conductance g_0 grows. The largest conductance for which the temperature dependent data are available in Ref. 17 is $g_0 = 2\pi^2$. Setting $\frac{\tilde{E}_C(T^*)}{T^*} = 0.3$ and using Eq. (21) we find that $\frac{T^*}{E_C} \approx 10^{-2}$. The temperature dependence of $E_C^* = 2\pi^2 \tilde{E}_C$ in Ref. 17 saturates roughly at the same temperature. At this temperature $\ln \frac{T_0}{T^*} \approx 1.6$ and quantitative comparison of our results with numerical results is not justified. Qualitatively however, at this temperature the results of Eq. (21) and of Ref. 7 are not very different.

Thus quantitative comparison of numerical results with the results of different analytical approaches, Refs. 7,8 and Eq. (3) requires numerical studies at much lower temperatures and larger conductances than those available at present.

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APPENDIX A: EVALUATION OF $\det \hat{A}(z,\tilde{\phi}_z)$

To obtain the metric tensor $\hat{A}(z,\tilde{\phi}_z)$ in Eq. (17) we express the variation of the field ϕ through the variables z and $\{\tilde{\varphi}_n\}$ with the aid of Eqs. (10) and (15),

$$\begin{aligned}\delta\phi &= \frac{\partial\phi_z}{\partial z}\delta z + \sum_{n>0}\tilde{\varphi}_n u^n \frac{\partial f}{\partial z}\delta z + \sum_{n<0}\tilde{\varphi}_n u^n \frac{\partial f^*}{\partial z}\delta z \\ &+ \frac{\partial\phi_z}{\partial z^*}\delta z^* + \sum_{n>0}\tilde{\varphi}_n u^n \frac{\partial f}{\partial z^*}\delta z^* + \sum_{n<0}\tilde{\varphi}_n u^n \frac{\partial f^*}{\partial z^*}\delta z^* \\ &+ \sum_{n>0}\delta\tilde{\varphi}_n u^n f(u) + \sum_{n<0}\delta\tilde{\varphi}_n u^n f^*(u). \qquad \text{(A1)}\end{aligned}$$

Below we omit the arguments z and $\tilde{\phi}_z$ of the metric tensor $\hat{A}$. Its matrix elements are obtained from the following relation,

$$\begin{aligned}\oint \frac{du}{2\pi i u}|\delta\phi|^2 &= \sum_{n,m} A_{nm}\delta\tilde{\varphi}_n^*\delta\tilde{\varphi}_m + 2A_{zz^*}\delta z\delta z^* \\ &+ A_{zz}^*\delta z^*\delta z^* + A_{zz}\delta z\delta z \\ &+ 2\sum_m \left[A_{zm}\delta z + A_{z^*m}\delta z^*\right]\delta\tilde{\varphi}_m. \text{(A2)}\end{aligned}$$

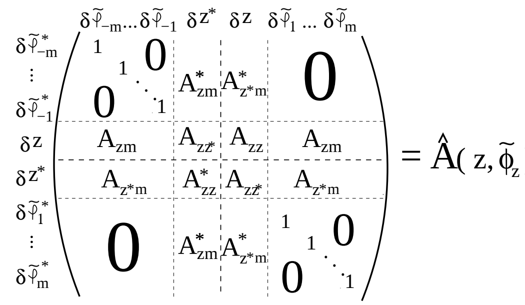

FIG. 3: Matrix elements of the metric tensor $\hat{A}(z,\tilde{\phi}_z)$ in coordinates z, $\{\tilde{\varphi}_n\}$.

Substituting Eq. (A1) into Eq. (A2) it is straightforward to find all the elements of the metric tensor $\hat{A}(z,\tilde{\phi}_z)$. Most of the matrix elements of $\hat{A}(z,\tilde{\phi}_z)$ vanish. The schematic form of this matrix is shown in Fig. 3.

The determinant of $\hat{A}(z,\tilde{\phi}_z)$ in Eq. (17) needs to be evaluated up to terms of order $1/g_0$. Therefore it is sufficient to find the matrix element A_{zz} up to the linear order in $\tilde{\varphi}$. Integrating over the variable u in Eq. (A2) we obtain the following expressions for the elements of matrix $\hat{A}(z,\tilde{\phi}_z)$,

$$A_{zz} = -i\sum_{n>0}\tilde{\varphi}_n \frac{z^{n-1}}{1-|z|^2} - i\sum_{n<0}\tilde{\varphi}_n^* \frac{z^{-n-1}}{1-|z|^2}, \qquad \text{(A3a)}$$

$$A_{zz^*} = \frac{1}{1-|z|^2}\left(1 + 2\sum_{n>0}|\tilde{\varphi}_n|^2\right), \qquad \text{(A3b)}$$

$$A_{zm} = \begin{cases} \sum\limits_{n>0}\tilde{\varphi}_n^* z^{m-n-1}\theta(m-n), & m>0, \\ -\sum\limits_{n<0}\tilde{\varphi}_n^* z^{m-n-1}\theta(m-n), & m<0, \end{cases} \qquad \text{(A3c)}$$

$$A_{z^*m} = \begin{cases} -\sum\limits_{n>0}\tilde{\varphi}_n^* (z^*)^{n-m-1}\theta(n-m), & m>0, \\ \sum\limits_{n<0}\tilde{\varphi}_n^* (z^*)^{n-m-1}\theta(n-m), & m<0. \end{cases} \qquad \text{(A3d)}$$

Here the θ-function should be understood as $\theta(0) = 0$. We note that even for very long instanton, $z \to 0$, some off-diagonal elements of matrix $\hat{A}(z,\tilde{\phi}_z)$ remain finite, see Eq. (A3).

To evaluate the renormalized charging energy $\tilde{E}_C$ in the Gaussian approximation, see Sec. III A, the Jacobian in Eq. (17) may be evaluated on the instanton trajectory, $\tilde{\phi}_z = 0$. In this case the determinant of $\hat{A}(z,\tilde{\phi}_z)$ is readily evaluated, and we obtain Eq. (18).

In order to obtain the leading $1/g_0$ correction to the Gaussian result for $\tilde{E}_C$, we need to calculate the square

root of the determinant of $\hat{A}(z,\tilde{\phi}_z)$ in Eq. (25) to second order in $\tilde{\phi}_z$. To this end we expand $\det\hat{A}(z,\tilde{\phi})$ to second order in its off-diagonal elements and perform the averaging with respect to the Gaussian fluctuations around the instanton, $\langle\ldots\rangle_i$, in Eq. (25). We show in Appendix B that the off-diagonal elements of the charging part of the action may be neglected, and therefore $\langle\tilde{\varphi}_n\tilde{\varphi}^*_m\rangle_i = \delta_{nm}\left(g_0|n| + \frac{2\pi^2 T}{E_C}\left[(|n|+1)^2 + \frac{2|z|^2}{1-|z|^2}\right]\right)^{-1}$. Using this result and Eq. (A3) we obtain,

$$\left\langle\sqrt{\det\hat{A}(z,\tilde{\phi})}\right\rangle_i = \frac{1}{1-|z|^2}\left(1+\frac{1}{g_0}\ln\frac{g_0 T_0}{T}\right). \quad \text{(A4)}$$

APPENDIX B: CHARGING PART OF THE ACTION

To obtain the charging energy, Eq. (27) we evaluated the functional integral over the massive fluctuations about the instanton in the basis of Eq. (15). We neglected the off-diagonal elements in the charging part of the action. In this appendix we show that this approximation is justified. Taking the off-diagonal matrix elements into account leads to a correction to $\tilde{E}_C$ which is smaller than the non-Gaussian correction given by the last two terms in Eq. (27) by a factor of $\ln\frac{E_C}{T}$. Therefore it can be neglected at low temperatures.

To demonstrate this we rewrite the charging part of the action using the complex variable $u = \exp(2\pi i T\tau)$ and Eq. (10),

$$\int_0^\beta \frac{\dot{\phi}^2(\tau)}{4E_C}d\tau = \frac{\pi i T}{2E_C}\oint u du$$
$$\times\left[\left(\frac{\partial\phi_z}{\partial u}\right)^2 + 2\frac{\partial\phi_z}{\partial u}\frac{\partial\tilde{\phi}_z}{\partial u} + \left(\frac{\partial\tilde{\phi}_z}{\partial u}\right)^2\right]. \quad \text{(B1)}$$

Here the instanton configuration ϕ_z was defined in Eq. (8). First term in the right hand side of Eq. (B1) describes the charging part of the action on the instanton configuration and the last two terms describe small fluctuations around the instanton. Setting $\tilde{\phi}_z = 0$ and integrating over the variable u in Eq. (B1) we obtain for the charging action on the instanton configuration,

$$\int_0^\beta \frac{\dot{\phi}_z^2(\tau)}{4E_C}d\tau = \frac{\pi^2 T}{E_C}\left(\frac{1+|z|^2}{1-|z|^2}\right). \quad \text{(B2)}$$

To calculate the second and the third terms in the right hand side of Eq. (B1) we expand the massive fluctuations $\tilde{\phi}_z$ using Eq. (15). Integrating over the variable u in Eq. (B1) we obtain the following results

$$S_2 \equiv \frac{\pi i T}{E_C}\oint\frac{\partial\phi_z}{\partial u}\frac{\partial\tilde{\phi}_z}{\partial u}u du = \frac{2\pi^2 i T}{E_C(1-|z|^2)}$$
$$\times\left(\sum_{n>0}\tilde{\varphi}_n z^{n+1} - \sum_{n<0}\tilde{\varphi}_n (z^*)^{-n+1}\right), \quad \text{(B3)}$$
$$S_3 \equiv \frac{\pi i T}{2E_C}\oint\left(\frac{\partial\tilde{\phi}_z}{\partial u}\right)^2 u du = \frac{\pi^2 T}{E_C}$$
$$\times\sum_{n,m>0}\left[\tilde{\varphi}_n f_{nm}(z)\tilde{\varphi}_{-m} + \tilde{\varphi}_m f_{mn}(z)\tilde{\varphi}_{-n}\right]. \quad \text{(B4)}$$

Here the function $f_{nm}(z)$ is given by the following expression

$$f_{nm}(z) = n^2\delta_{n,m}$$
$$+(m+n)\left[z^{n-m}\theta(n-m+1) + (z^*)^{m-n}\theta(m-n)\right]$$
$$+z^{n-m}\left[n-m+\frac{1+|z|^2}{1-|z|^2}\right]\theta(n-m+2)$$
$$+(z^*)^{m-n}\left[m-n+\frac{1+|z|^2}{1-|z|^2}\right]\theta(m-n-1). \quad \text{(B5)}$$

Here again $\theta(0) = 0$. From Eqs. (B4) and (B5) it follows that the diagonal part of the quadratic form of the charging action is given by the following expression,

$$S_3^{diag} = \frac{2\pi^2 T}{E_C}\sum_{n>0}|\tilde{\varphi}_n|^2\left[(n+1)^2 + \frac{2|z|^2}{1-|z|^2}\right]. \quad \text{(B6)}$$

The result of Eq. (B6) was used in Eq. (19).

We now consider the contribution to the renormalized charging energy $\tilde{E}_C(T)$ in Eq. (27) from the term S_2 in Eq. (B3). We expand the numerator in the right hand side of Eq. (12) to second order in S_2. As a result we obtain the following correction to the right hand side of Eq. (20),

$$\delta Z_1^{\mathrm{G}}(z) = \frac{g_0^2 E_C}{2\pi^3 T}\frac{\langle S_2^2\rangle_i}{2}\exp\left[-\frac{g_0}{2} - \frac{\pi^2 T[1+|z|^2]}{E_C[1-|z|^2]}\right]. \quad \text{(B7)}$$

Here the angular brackets, $\langle\ldots\rangle_i$, denote averaging with respect to the Gaussian action, Eq. (16). From Eq. (B3) we obtain,

$$\frac{\langle S_2^2\rangle_i}{2} = \left(\frac{2\pi^2 T}{E_C[1-|z|^2]}\right)^2\sum_{n>0}|z|^{2(n+1)}\left\langle|\tilde{\varphi}_n|^2\right\rangle_i. \quad \text{(B8)}$$

Using the fact that $\left\langle|\tilde{\varphi}_n|^2\right\rangle_i = (g_0|n|)^{-1}$ and substituting Eqs. (B7), (B8) into Eq. (11) after integration over the z we obtain the following correction $\delta\tilde{E}_C(T)$ to the renormalized charging energy $\tilde{E}_C(T)$ in Eq. (27)

$$\delta\tilde{E}_C(T) = \frac{g_0^2 E_C}{\pi^2}\left[\frac{\ln\frac{T_0}{T}}{g_0}\right]\exp\left(-\frac{g_0}{2}\right). \quad \text{(B9)}$$

This correction is proportional to the first power of the large logarithm $\ln(T_0/T)$. Therefore it is much smaller than the second term in the right hand side of Eq. (27) and can be neglected. It is straightforward to check that the contribution to the renormalized charging energy $\tilde{E}_C(T)$ in Eq. (27) from the term S_3 in Eq. (B4) is small as $1/g_0^2$ in comparison with the result of Eq. (B9) and therefore can also be neglected.

APPENDIX C: EVALUATION OF THE LAST THREE TERMS IN EQ. (25)

We employ Wick's theorem to rewrite the last three averages in Eq. (25) in the following form,

$$\begin{aligned}
&\left\langle(\tilde{\phi}-\tilde{\phi}_1)^4\right\rangle = 3\left\langle(\tilde{\phi}-\tilde{\phi}_1)^2\right\rangle_i^2,\\
&\left\langle(\tilde{\phi}-\tilde{\phi}_1)^3(\tilde{\phi}_2-\tilde{\phi}_3)^3\right\rangle_i = 6\left\langle(\tilde{\phi}-\tilde{\phi}_1)(\tilde{\phi}_2-\tilde{\phi}_3)\right\rangle_i^3\\
&+9\left\langle(\tilde{\phi}-\tilde{\phi}_1)(\tilde{\phi}_2-\tilde{\phi}_3)\right\rangle_i\left\langle(\tilde{\phi}-\tilde{\phi}_1)^2\right\rangle_i\left\langle(\tilde{\phi}_2-\tilde{\phi}_3)^2\right\rangle_i,\\
&\left\langle(\phi-\phi_1)^4\right\rangle_0 = 3\left\langle(\phi-\phi_1)^2\right\rangle_0^2.
\end{aligned}$$

Different pairings in this equation can be represented by diagrams in Fig. 1.

As is verified by the subsequent calculations, only fluctuations of ϕ with frequencies below $T/(1-|z|^2)$ contribute to the corresponding functional integrals. Therefore we neglect the charging energy term in the Gaussian action for the massive fluctuations of ϕ. It is then convenient to perform the calculations in the bases (13), (15) which diagonalize[6] the dissipative action. For example, for the correlator $\langle(\tilde{\phi}-\tilde{\phi}_1)(\tilde{\phi}_2-\tilde{\phi}_3)\rangle_i$ we obtain the following expression,

$$\left\langle(\tilde{\phi}-\tilde{\phi}_1)(\tilde{\phi}_2-\tilde{\phi}_3)\right\rangle_i = -2\sum_{n\geq 1}\langle|\tilde{\varphi}_n|^2\rangle_i \times\mathrm{Re}\left[\frac{u_1 f(u_1)}{u_2 f(u_2)}h_n(u,u_1)h_n(u_2,u_3)\right]. \tag{C1}$$

Here we introduced the notation

$$h_n(u,u_1) = 1-\left[\frac{u}{u_1}\right]^n\frac{f(u)}{f(u_1)}. \tag{C2}$$

All other correlation functions may be expressed through $\langle|\tilde{\varphi}_n|^2\rangle_i$ and $\langle|\varphi_n|^2\rangle_0$ in a similar way. Since upon neglecting the charging action $\langle|\tilde{\varphi}_n|^2\rangle_i = \langle|\varphi_n|^2\rangle_0 = (g_0|n|)^{-1}$ we can write the averages in the right hand side of Eq. (25) in the following form:

$$\left\langle S^i_{\tilde{\phi}^4}\right\rangle_i = \frac{2}{g_0}\sum_{n,n_1\geq 1}^{\infty}\frac{1}{nn_1}\oint\frac{du\,du_1}{(2\pi i)^2(u-u_1)^2}\mathrm{Re}\left[\frac{f(u)}{f(u_1)}\right]\mathrm{Re}\left[h_n(u,u_1)\right]\mathrm{Re}\left[h_{n_1}(u,u_1)\right], \tag{C3a}$$

$$\left\langle S^0_{\phi^4}\right\rangle_0 = \frac{2}{g_0}\sum_{n,n_1\geq 2}^{\infty}\frac{1}{nn_1}\oint\frac{du\,du_1}{(2\pi i)^2(u-u_1)^2}\mathrm{Re}\left[1-\left[\frac{u}{u_1}\right]^n\right]\mathrm{Re}\left[1-\left[\frac{u}{u_1}\right]^{n_1}\right], \tag{C3b}$$

$$\begin{aligned}\frac{\left\langle(S^i_{\tilde{\phi}^3})^2\right\rangle_i}{2} &= -\frac{2}{3g_0}\sum_{n,n_1,n_2\geq 1}^{\infty}\frac{1}{nn_1n_2}\oint\frac{du\,du_1du_2du_3}{(2\pi)^4(u-u_1)^2(u_2-u_3)^2}\mathrm{Im}\left[\frac{f(u)}{f(u_1)}\right]\\
&\times\Big(6\mathrm{Re}[h_{n_1}(u,u_1)]\mathrm{Re}[h_{n_2}(u_2,u_3)]\mathrm{Re}[y_n]+\mathrm{Re}[y_n]\mathrm{Re}[y_{n_1}]\mathrm{Re}[y_{n_2}]\Big),\end{aligned} \tag{C3c}$$

where y_{n_i} denotes the following function,

$$y_{n_i} = \left[\frac{u_1}{u_2}\right]^{n_i}\frac{f(u_1)}{f(u_2)}h_{n_i}(u,u_1)h_{n_i}(u_2,u_3). \tag{C4}$$

The two terms in the last line of Eq. (C3c) correspond to the second and third diagrams in Fig. 1 respectively.

Next we perform the integration over u_i in Eq. (C3). The first term in the last line of Eq. (C3c) vanishes. The remaining integrals lead to the results:

$$\left\langle S^i_{\tilde{\phi}^4}\right\rangle_i = -\frac{1}{2g_0}\sum_{n,n_1\geq 1}^{\infty}\frac{2\min(\mathrm{n},\mathrm{n}_1)-|z|^{2|n-n_1|}}{nn_1}, \tag{C5a}$$

$$\begin{aligned}\frac{\left\langle(S^i_{\tilde{\phi}^3})^2\right\rangle_i}{2} &= -\frac{[1-|z|^2]^2}{6g_0}\sum_{n,n_1,n_2\geq 1}^{\infty}\frac{1}{nn_1n_2}\\
&\times[2(n_2-n-n_1)^2|z|^{2(n_2-n_1-n-1)}\theta(n_2-n-n_1)\\
&+(n-n_2-n_1)^2|z|^{2(n-n_1-n_2-1)}\theta(n-n_2-n_1)],\end{aligned} \tag{C5b}$$

$$\left\langle S^0_{\phi^4}\right\rangle_0 = -\frac{1}{g_0}\sum_{n,n_1\geq 1}^{\infty}\frac{\min(\mathrm{n},\mathrm{n}_1)}{nn_1}. \tag{C5c}$$

Evaluating the sums in Eq. (C5) with logarithmic accuracy and using Eq. (25) we obtain Eq. (26).

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